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# Using rigged configurations to model $B(\infty)$

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**Abstract.** Crystal bases provide a rich environment for one to study quantized universal enveloping algebras and their representation theory for any symmetrizable Kac-Moody algebra by elucidating the underlying combinatorics. While the definition of a crystal basis involves complicated algebra, the combinatorial nature allows these crystals to be modeled using combinatorial objects. In this work, the underlying combinatorial model consists of rigged configurations, which allow for a uniform description of these crystals across all symmetrizable Kac-Moody types. Their flexibility is exhibited by the fact that the combinatorial isomorphism to crystals of tableaux is understood and that the star-crystal structure is easily computable directly from the rigged configurations. These results are summarized in this abstract.

**Keywords:** rigged configuration, crystal, star-involution, tableau, quantum group

## 1 Introduction

Crystal basis theory is an elegant and fruitful subject born out of the theory of quantum groups. Defined by Kashiwara in the early 1990s [10], crystals provide a natural combinatorial framework to study the representations of Kac-Moody algebras (including classical Lie algebras) and their associated quantum groups. Their applications span many areas of mathematics, including representation theory, algebraic combinatorics, automorphic forms, and mathematical physics, to name a few.

The study of crystal bases has led researchers to develop different combinatorial models for crystals which yield suitable settings to studying a particular aspect of the representation theory of quantum groups. For example, highest weight crystals (which are combinatorial skeletons of an irreducible highest weight module over a quantum group) can be modeled using several different combinatorial, algebraic, or geometric objects. The choice of using one model over the other usually depends on the underlying question at hand (and/or on the preference of the author). In concert with the descriptions for the highest weight crystals, there are several known realizations of the (infinite) crystal $B(\infty)$ (which is a combinatorial skeleton for the Verma module with

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highest weight 0), both in combinatorial and geometric settings, which have various applications.

Our choice of model will be that of *rigged configurations*, which arise naturally as indexing the eigenvalues and eigenvectors of a Hamiltonian of a statistical model [2, 14, 15]. On the other hand, these eigenvectors may also be indexed by one-dimensional lattice paths [1, 8, 7, 19], which can be interpreted as highest weight vectors in a tensor product of certain crystals. In recent years, the implied connection between highest weight vectors in tensor products of Kirillov-Reshetikhin crystals and rigged configurations has been worked out [20, 21, 24, 30, 32].

As was shown in [25] and explained in Section 3 below, the rigged configuration model has simple combinatorial rules for describing the structure which work in all finite, affine, and all simply-laced Kac-Moody types. These combinatorial rules are only based on the nodes of the Dynkin diagram and their neighbors. This allows us to easily describe the morphisms of $B(\lambda)$ into $B(\mu)$ (up to a weight shift). Moreover, we can easily describe the so-called virtualization of $B(\lambda)$ inside of a highest weight crystal of another type via a diagram folding [21, 22, 31]. Extending the proof general symmetrizable Kac-Moody types was completed in [28] using a generalization of Lusztig's admissible folding technique [18]. In Section 5, we describe the model for the irreducible highest weight crystals obtained from this model for $B(\infty)$.

The $*$-involution is an involution on the crystal $B(\infty)$ that is induced from a subtle involutive antiautomorphism of $U_q(\mathfrak{g})$. The importance of $*$ in the theory of crystal bases and their applications cannot be understated. Several combinatorial realizations of the $*$-involution are known in the literature. Indeed, model-specific calculations of the $*$-crystal operators are important as, *a priori*, the algorithm for computing the action of these operators is not efficient [11, Thm. 2.2.1] (see also [12, Prop. 8.1]). In Section 4, a description of the $*$-involution on $\mathrm{RC}(\infty)$ is given, as well as a combinatorial description of the $*$-crystal operators on $\mathrm{RC}(\infty)$.

In [14, 15], Kerov, Kirillov, and Reshetikhin described a recursive bijection $\Phi$ between classically highest-weight rigged configurations in type $A_n^{(1)}$ and standard Young tableaux, showing the Kostka polynomial can be expressed as a fermionic formula. The bijection $\Phi$ was then extended to Littlewood-Richardson tableaux and classically highest weight elements in a tensor product of Kirillov-Reshetikhin (KR) crystals in [16] for, again, type $A_n^{(1)}$. This has been generalized to a number of other special cases (see, *e.g.*, [22, 23, 29, 31, 33] and references therein).

The description of $\Phi$ on classically highest-weight elements led to a description of classical crystal operators in simply-laced types in [30] and non-simply-laced finite types in [31]. It was shown for type $A_n^{(1)}$ in [4] and $D_n^{(1)}$ in [24] that $\Phi$ is a classical crystal isomorphism. Using virtual crystals [21], it can be shown $\Phi$ is a classical crystal isomorphism in non-exceptional affine types [31]. In Section 6, a crystal isomorphism between

$\mathrm{RC}(\infty)$ and the marginally large tableaux model is exhibited by extending the bijection $\Phi$. In particular, the crystal isomorphism is given combinatorially, in the sense that the description does not use the Kashiwara operators.

## 2 Abstract crystals

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra with quantized universal enveloping algebra $U_q(\mathfrak{g})$ over $\mathbf{Q}(q)$, index set $I$, generalized Cartan matrix $A = (A_{ab})_{a,b\in I}$, weight lattice $P$, root lattice $Q$, fundamental weights $\{\Lambda_a : a \in I\}$, simple roots $\{\alpha_a : a \in I\}$, and simple coroots $\{h_a : a \in I\}$. There is a canonical pairing $\langle\ ,\ \rangle \colon P^\vee \times P \longrightarrow \mathbf{Z}$ defined by $\langle h_a, \alpha_b\rangle = A_{ab}$, where $P^\vee$ is the dual weight lattice.

An *abstract $U_q(\mathfrak{g})$-crystal* is a set $B$ together with maps $e_a, f_a \colon B \longrightarrow B \sqcup \{0\}$ (*Kashiwara operators*), $\varepsilon_a, \varphi_a \colon B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}$, and a *weight map* $\mathrm{wt}\colon B \longrightarrow P$ satisfying certain conditions (see, for example, [12]). Any $U_q(\mathfrak{g})$-crystal basis, defined in the classical sense (see [10]), is an abstract $U_q(\mathfrak{g})$-crystal. In particular, the negative half $U_q^-(\mathfrak{g})$ of the quantized universal enveloping algebra of $\mathfrak{g}$ has a crystal basis which is an abstract $U_q(\mathfrak{g})$-crystal. We denote this crystal by $B(\infty)$ (rather than the using the entire tuple $(B(\infty), e_a, f_a, \varepsilon_a, \varphi_a, \mathrm{wt})$), and denote its highest weight element by $u_\infty$. As a set, one has

$$B(\infty) = \{f_{a_d} \cdots f_{a_2} f_{a_1} u_\infty : a_1, \dots, a_d \in I,\ d \geq 0\}.$$

The remaining crystal structure on $B(\infty)$ is

$$\mathrm{wt}(f_{a_d} \cdots f_{a_2} f_{a_1} u_\infty) = -\alpha_{a_1} - \alpha_{a_2} - \cdots - \alpha_{a_d},$$
$$\varepsilon_a(v) = \max\{k \in \mathbf{Z} : e_a^k v \neq 0\}, \quad \varphi_a(v) = \varepsilon_a(v) + \langle h_a, \mathrm{wt}(v)\rangle.$$

We say that $v \in B(\infty)$ has *depth $d$* if $v = f_{a_d} \cdots f_{a_2} f_{a_1} u_\infty$ for some $a_1, \dots, a_d \in I$.

A *crystal isomorphism* $B_1 \cong B_2$ between two abstract $U_q(\mathfrak{g})$-crystals is a bijection between $B_1 \sqcup \{0\}$ and $B_2 \sqcup \{0\}$ that commutes with $e_a$ and $f_a$, for all $a \in I$, and preserves the weight.

Again, let $B_1$ and $B_2$ be abstract $U_q(\mathfrak{g})$-crystals. The *tensor product* of crystals $B_2 \otimes B_1$ is $B_2 \times B_1$ as a set, endowed with the following crystal structure. The Kashiwara operators are given by

$$e_a(v_2 \otimes v_1) = \begin{cases} e_a v_2 \otimes v_1 & \varepsilon_a(v_2) > \varphi_a(v_1), \\ v_2 \otimes e_a v_1 & \text{otherwise}, \end{cases} \quad f_a(v_2 \otimes v_1) = \begin{cases} f_a v_2 \otimes v_1 & \varepsilon_a(v_2) \geq \varphi_a(v_1), \\ v_2 \otimes f_a v_1 & \text{otherwise}. \end{cases}$$

The remainder of the crystal structure is determined by setting $\mathrm{wt}(v_2 \otimes v_1) = \mathrm{wt}(v_2) + \mathrm{wt}(v_1)$. Note that this is opposite to Kashiwara's convention [10] for the tensor product.

There is a $\mathbf{Q}(q)$-antiautomorphism $*\colon U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})$ defined by $E_a^* = E_a$, $F_a^* = F_a$, $q^* = q$, and $(q^h)^* = q^{-h}$. This is an involution which leaves $U_q^-(\mathfrak{g})$ stable. Thus, the map

$*$ induces a map on $B(\infty)$, which we also denote by $*$, and is called the $*$-*involution* or *star involution* (and is sometimes known as Kashiwara's involution). Denote the image of $B(\infty)$ under $*$ by $B(\infty)^*$.

**Theorem 2.1** ([11, 17]). *We have $B(\infty)^* = B(\infty)$.*

This induces a new crystal structure on $B(\infty)$ with *star-crystal operators* $e_a^* = * \circ e_a \circ *$ and $f_a^* = * \circ f_a \circ *$, and the remaining crystal structure is given by $\varepsilon_a^* = \varepsilon_a \circ *$, $\varphi_a^* = \varphi_a \circ *$, and weight function wt, the usual weight function on $B(\infty)$. Additionally, for $v \in B(\infty)$ and $a \in I$, define $\kappa_a(v) := \varepsilon_a(v) + \varepsilon_a^*(v) + \langle h_a, \mathrm{wt}(v) \rangle$.

We will appeal to a statement from [34], which is based on the classification theorem for $B(\infty)$ from [13]. First, a *bicrystal* is a set $B$ with two abstract $U_q(\mathfrak{g})$-crystal structures $(B, e_a, f_a, \varepsilon_a, \varphi_a, \mathrm{wt})$ and $(B, e_a^\star, f_a^\star, \varepsilon_a^\star, \varphi_a^\star, \mathrm{wt})$ with the same weight function. In such a bicrystal $B$, we say $v \in B$ is a *highest weight element* if $e_a v = e_a^\star v = 0$ for all $a \in I$. The following is a minor weakening of the assumptions in [34].

**Proposition 2.2.** *Let $(B, e_a, f_a, \varepsilon_a, \varphi_a, \mathrm{wt})$ and $(B^\star, e_a^\star, f_a^\star, \varepsilon_a^\star, \varphi_a^\star, \mathrm{wt})$ be abstract $U_q(\mathfrak{g})$-crystals with the same highest weight vector $v_0 \in B \cap B^\star$, where the remaining crystal data is determined by setting $\mathrm{wt}(v_0) = 0$. Assume further that, for all $a \neq b$ in $I$ and all $v \in B$,*

*(1) $f_a v,\ f_a^\star v \neq 0$;*

*(2) $f_a^\star f_b v = f_b f_a^\star v$;*

*(3) $\kappa_a(v) \geq 0$;*

*(4) $\kappa_a(v) = 0$ implies $f_a v = f_a^\star v$;*

*(5) $\kappa_a(v) \geq 1$ implies $\varepsilon_a^\star(f_a v) = \varepsilon_a^\star(v)$ and $\varepsilon_a(f_a^\star v) = \varepsilon_a(v)$;*

*(6) $\kappa_a(v) \geq 2$ implies $f_a f_a^\star v = f_a^\star f_a v$.*

*Then $(B, e_a, f_a, \varepsilon_a, \varphi_a, \mathrm{wt}) \cong (B^\star, e_a^\star, f_a^\star, \varepsilon_a^\star, \varphi_a^\star, \mathrm{wt}) \cong B(\infty)$, with $B = B^\star$, $e_a^\star = e_a^*$, $f_a^\star = f_a^*$.*

# 3 Rigged configurations and $\mathrm{RC}(\infty)$

Let $\mathcal{H} = I \times \mathbf{Z}_{>0}$. A rigged configuration is a sequence of partitions $\nu = (\nu^{(a)} : a \in I)$ such that each row $\nu_i^{(a)}$ has an integer called a *rigging*, and we let $J = \big(J_i^{(a)} : (a,i) \in \mathcal{H}\big)$, where $J_i^{(a)}$ is the multiset of riggings of rows of length $i$ in $\nu^{(a)}$. If $x \in J_i^{(a)}$, the pair $(i, x)$ is called a *string*. We consider there to be an infinite number of rows of length 0 with rigging 0; i.e., $J_0^{(a)} = \{0, 0, \dots\}$ for all $a \in I$. The term rigging will be interchanged freely with the term *label*. We identify two rigged configurations $(\nu, J)$ and $(\widetilde{\nu}, \widetilde{J})$ if $\nu = \widetilde{\nu}$ and $J_i^{(a)} = \widetilde{J}_i^{(a)}$ for any fixed $(a,i) \in \mathcal{H}$. Let $(\nu, J)^{(a)}$ denote the rigged partition $(\nu^{(a)}, J^{(a)})$. Define the *vacancy numbers* of $\nu$ to be

$$p_i^{(a)}(\nu) = p_i^{(a)} = -\sum_{(b,j) \in \mathcal{H}} A_{ab} \min(i,j) m_j^{(b)}, \tag{3.1}$$

where $m_i^{(a)}$ is the number of parts of length $i$ in $\nu^{(a)}$. The *corigging*, or *colabel*, of a row in $(\nu, J)^{(a)}$ with rigging $x$ is $p_i^{(a)} - x$. A string $(i, x)$ is called *singular* if $x = p_i^{(a)}$ and is called *quasisingular* if $x = p_i^{(a)} - 1$.

Let $\mathrm{RC}(\infty)$ denote the set of rigged configurations generated by $(\nu_\emptyset, J_\emptyset)$, where $\nu_\emptyset^{(a)} = 0$ for all $a \in I$, and closed under the crystal operators as follows.

**Definition 3.2** ([25, 30])**.** *Fix some $a \in I$, and let $x$ be the smallest rigging in $(\nu, J)^{(a)}$.*

- $e_a$: *If $x = 0$, then $e_a(\nu, J) = 0$. Otherwise, let $r$ be a row in $(\nu, J)^{(a)}$ of minimal length $\ell$ with rigging $x$. Then $e_a(\nu, J)$ is the rigged configuration which removes a box from row $r$, sets the new rigging of $r$ to be $x + 1$, and changes all other riggings such that the coriggings remain fixed.*
- $f_a$: *Let $r$ be a row in $(\nu, J)^{(a)}$ of maximal length $\ell$ with rigging $x$. Then $f_a(\nu, J)$ is the rigged configuration which adds a box to row $r$, sets the new rigging of $r$ to be $x - 1$, and changes all other riggings such that the coriggings remain fixed.*

We define the remainder of the crystal structure on $\mathrm{RC}(\infty)$ by

$$\varepsilon_a(\nu, J) = \max\{k \in \mathbf{Z} : e_a^k(\nu, J) \neq 0\}, \qquad \mathrm{wt}(\nu, J) = -\sum_{a \in I} |\nu^{(a)}| \alpha_a,$$

and $\varphi_a(\nu, J) = \langle h_a, \mathrm{wt}(\nu, J) \rangle + \varepsilon_a(\nu, J)$.

**Theorem 3.3** ([25, 28])**.** *Let $\mathfrak{g}$ be of symmetrizable type. Then $\mathrm{RC}(\infty) \cong B(\infty)$.*

The proof of Theorem 3.3 was proved in a series of steps. The first step appealed to Schilling's work in [30] using crystal operators that differ slightly from those in Definition 3.2. The operators in [30], together with a suitable highest weight rigged configuration, form a model for irreducible highest weight crystals $B(\lambda)$ in simply-laced types. An argument similar to that used in [24] showed that $\mathrm{RC}(\infty)$ forms an abstract crystal in simply-laced types. Then, using the fact that $B(\infty)$ is the direct limit of the $B(\lambda)$, $\mathrm{RC}(\infty)$ is shown to be isomorphic to $B(\infty)$ in any simply-laced type. Subsequently, using the theory of virtual crystals and a recognition theorem of Kashiwara and Saito [13], $\mathrm{RC}(\infty)$ was shown to be isomorphic to $B(\infty)$ in all finite and all affine types. However, to obtain the desired result for any symmetrizable types, one needs to modify the method of admissible foldings, due to Lusztig [18], so that any symmetrizable type may be obtained from a simply-laced Kac-Moody algebra (as opposed to a symmetric Kac-Moody algebra). This relaxation is the content of [28] and completed the characterization of $B(\infty)$ in terms of rigged configurations for all symmetrizable Kac-Moody types.

# 4 Star-crystal structure

**Definition 4.1** ([27])**.** *Fix some $a \in I$, and let $x$ be the smallest* corigging *in $(\nu, J)^{(a)}$.*

$e_a^*$: *If $x = 0$, then $e_a^*(\nu, J) = 0$. Otherwise let $r$ be a row in $(\nu, J)^{(a)}$ of minimal length $\ell$ with corigging $x$. Then $e_a^*(\nu, J)$ is the rigged configuration which removes a box from row $r$ and sets the new corigging of $r$ to be $x + 1$.*

$f_a^*$: *Let $r$ be a row in $(\nu, J)^{(a)}$ of maximal length $\ell$ with corigging $x$. Then $f_a^*(\nu, J)$ is the rigged configuration which adds a box to row $r$ and sets the new colabel of $r$ to be $x - 1$.*

**Example 4.2.** *Consider type $D_4$, where $\alpha_2$ is the trivalent node in the Dynkin diagram. Set*

$$(\nu, J) = \; -1\ \square\square\ 0 \quad \begin{array}{rl} -3 & \square\square\square\ -2 \\ -1 & \square\ -1 \end{array} \quad -1\ \square\square\ 0 \quad 0\ \square\ 0\ .$$

*Then* $f_2^*(\nu, J) = \; -1\ \square\square\ 0 \quad \begin{array}{rl} -5 & \square\square\square\square\ -3 \\ -1 & \square\ -1 \end{array} \quad -1\ \square\square\ 0 \quad 0\ \square\ 0\ .$

Let $\mathrm{RC}(\infty)^*$ denote the closure of $(\nu_\emptyset, J_\emptyset)$ under $f_a^*$ and $e_a^*$. The remaining crystal structure is

$$\varepsilon_a^*(\nu, J) = \max\{k \in \mathbf{Z} : (e_a^*)^k(\nu, J) \neq 0\}, \qquad \mathrm{wt}(\nu, J) = -\sum_{a \in I} |\nu^{(a)}| \alpha_a,$$

and $\varphi_a^*(\nu, J) = \langle h_a, \mathrm{wt}(\nu, J) \rangle + \varepsilon_a^*(\nu, J)$.

**Example 4.3.** *Let $(\nu, J)$ be the rigged configuration from Example 4.2. Then $\kappa_2(\nu, J) = 0$, and*

$$f_2(\nu, J) = \; -1\ \square\square\ 0 \qquad \begin{array}{rl} -5 & \square\square\square\square\ -3 \\ -1 & \square\ -1 \end{array} \qquad -1\ \square\square\ 0 \qquad 0\ \square\ 0\ .$$

*One can check that this agrees with $f_2^*(\nu, J)$ from Example 4.2, thereby illustrating an instance where Proposition 2.2(4) holds. Moreover, $\varepsilon_3(\nu, J) = 0$, $\varepsilon_3^*(\nu, J) = 1$, and $\kappa_3(\nu, J) = 1$. We have*

$$f_3(\nu, J) = \; -1\ \square\square\ 0 \qquad \begin{array}{rl} -1 & \square\square\square\ -1 \\ -1 & \square\ -1 \end{array} \qquad -2\ \square\square\square\ -1 \qquad 0\ \square\ 0$$

$$f_3^*(\nu, J) = \; -1\ \square\square\ 0 \qquad \begin{array}{rl} -2 & \square\square\square\ -2 \\ -1 & \square\ -1 \end{array} \qquad -2\ \square\square\square\ 0 \qquad 0\ \square\ 0\ .$$

*This illustrates an instance where Proposition 2.2(5) holds. Finally, $\kappa_4(\nu, J) = 2$ and an instance where Proposition 2.2(6) holds is*

$$f_4^* f_4(\nu, J) = f_4 f_4^*(\nu, J) = \; -1\ \square\square\ 0 \quad \begin{array}{rl} -1 & \square\square\square\ -1 \\ -1 & \square\ -1 \end{array} \quad -1\ \square\square\ 0 \quad -2\ \square\square\square\ -1\ .$$

**Theorem 4.4** ([27])**.** *Let $e_a$ and $f_a$ be the crystal operators given by Definition 3.2, and let $e_a^*$ and $f_a^*$ be given by Definition 4.1. Then $\big(\mathrm{RC}(\infty), e_a, f_a, \varepsilon_a, \varphi_a, \mathrm{wt}\big)$ and $\big(\mathrm{RC}(\infty)^*, e_a^*, f_a^*, \varepsilon_a^*, \varphi_a^*, \mathrm{wt}\big)$ satisfy the conditions of Proposition 2.2, so $\mathrm{RC}(\infty) = \mathrm{RC}(\infty)^*$, and $e_a^* = * \circ e_a \circ *$ and $f_a^* = * \circ f_a \circ *$ for all $a \in I$.*

**Corollary 4.5** ([27])**.** *The $*$-involution on $\mathrm{RC}(\infty)$ is given by replacing every rigging $x$ of a row of length $i$ in $(\nu, J)^{(a)}$ by the corresponding corigging $p_i^{(a)} - x$ for all $(a, i) \in \mathcal{H}$.*

# 5 Highest weight crystals

Let $P^+$ denote the set of dominant integral weights for $\mathfrak{g}$. For any $\lambda \in P^+$, define

$$\mathrm{RC}(\lambda) := \{(\nu, J) \in \mathrm{RC}(\infty) : \max J_i^{(a)} \leq p_i^{(a)}(\nu; \lambda) \text{ for all } (a,i) \in \mathcal{H}\},$$

where

$$p_i^{(a)}(\nu; \lambda) := \langle h_a, \lambda \rangle - \sum_{b \in I} A_{ab} \sum_{j \in \mathbf{Z}_{>0}} \min(i,j) m_j^{(b)}. \tag{5.1}$$

Note that Equation (5.1) differs from Equation (3.1) by $p_i^{(a)}(\nu) + \langle h_a, \lambda \rangle = p_i^{(a)}(\nu; \lambda)$.

We consider a crystal structure on $\mathrm{RC}(\lambda)$ as that inherited from $\mathrm{RC}(\infty)$ under the natural projection except with $\mathrm{wt}(\nu, J) = \lambda - \sum_{a \in I} |\nu^{(a)}| \alpha_a$.

**Theorem 5.2** ([25, 28, 30])**.** *We have* $\mathrm{RC}(\lambda) \cong B(\lambda)$.

Using the $*$-crystal structure, we easily obtain [12, Prop. 8.2].

**Proposition 5.3.** *Let $\lambda \in P^+$ and define $T_\lambda = \{t_\lambda\}$ to be the one-element crystal with operations defined as $e_a t_\lambda = f_a t_\lambda = 0$, $\varepsilon_a(t_\lambda) = \varphi_a(t_\lambda) = -\infty$, and $\mathrm{wt}(t_\lambda) = \lambda$, for all $a \in I$. Then*

$$\mathrm{RC}(\lambda) \cong \{t_\lambda \otimes (\nu, J) \in T_\lambda \otimes \mathrm{RC}(\infty) : \varepsilon_a^*(\nu, J) \leq \langle h_a, \lambda \rangle \textit{ for all } a \in I\}.$$

# 6 Connecting $\mathrm{RC}(\infty)$ to marginally large tableaux

Following [9], a semistandard tableau is called *marginally large* if the difference of the number of boxes in the $i$th row containing the element $i$ and the total number of boxes in the $(i+1)$st row is exactly 1. Such tableaux are defined for simple Lie algebras $\mathfrak{g}$ of type $A_n$, $B_n$, $C_n$, $D_{n+1}$, and $G_2$ in [9].

The set of marginally large tableaux may be generated through successive application of the Kashiwara lowering operators $f_a$ ($a \in I$) to a specified highest weight vector. It is in this way that the set of marginally large tableaux work as a combinatorial model for $B(\infty)$. In certain types, additional conditions are required to precisely define the model.

**Definition 6.1** ([9])**.** *For $X_n = A_n, B_n, C_n, D_{n+1}, G_2$, define the set $\mathcal{T}(\infty)$ as follows. (By convention, we assume $n = 2$ when $X_n = G_2$.) Each tableau $T \in \mathcal{T}(\infty)$ is marginally large and semistandard (with respect to the alphabet $J(X_n)$ in Figure 6.1) such that*

- *elements of the $i$th row (from the top, in English notation) are $\preceq \bar{\imath}$ in types $B_n$, $C_n$, and $D_{n+1}$ (and less than 3 in the second row in type $G_2$),*
- *a 0-box may occur at most once in a given row in type $B_n$ and at most once in the first row of a tableau in type $G_2$, and*
- *both $n+1$ and $\overline{n+1}$ may not appear in the same row in type $D_{n+1}$.*

| $X_n$ | $\mathcal{T}(\Lambda_1)$ | $J(X_n)$ |
|---|---|---|
| $A_n:$ | $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{n+1}$ | $\{1 \prec 2 \prec \cdots \prec n+1\}$ |
| $B_n:$ | $\boxed{1} \xrightarrow{1} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{1} \boxed{\overline{1}}$ | $\{1 \prec \cdots \prec n \prec 0 \prec \overline{n} \prec \cdots \prec \overline{1}\}$ |
| $C_n:$ | $\boxed{1} \xrightarrow{1} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\overline{1}}$ | $\{1 \prec \cdots \prec n \prec \overline{n} \prec \cdots \prec \overline{1}\}$ |
| $D_{n+1}:$ | $\begin{tikzcd} & & & \boxed{n+1} \arrow[dr, "n+1"] & & & \\ \boxed{1} \arrow[r, "1"] & \cdots \arrow[r, "n-1"] & \boxed{n} \arrow[ur, "n"] \arrow[dr, "n+1"'] & & \boxed{\overline{n}} \arrow[r, "n-1"] & \cdots \arrow[r, "1"] & \boxed{\overline{1}} \\ & & & \boxed{\overline{n+1}} \arrow[ur, "n"'] & & & \end{tikzcd}$ | $\left\{1 \prec \cdots \prec n \prec \begin{smallmatrix} n+1 \\ \overline{n+1} \end{smallmatrix} \prec \overline{n} \prec \cdots \prec \overline{1}\right\}$ |
| $G_2:$ | $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{1} \boxed{0} \xrightarrow{1} \boxed{\overline{3}} \xrightarrow{2} \boxed{\overline{2}} \xrightarrow{1} \boxed{\overline{1}}$ | $\{1 \prec 2 \prec 3 \prec 0 \prec \overline{3} \prec \overline{2} \prec \overline{1}\}$ |

**Figure 6.1:** The fundamental crystals $\mathcal{T}(\Lambda_1)$ and alphabets $J(X_n)$, for $X_n = A_n$, $B_n$, $C_n$, $D_{n+1}$, and $G_2$.

The crystal operators are defined by reading entries of a tableau $T \in \mathcal{T}(\infty)$ from top-to-bottom in columns starting with the right-most column to obtain an element of $\mathcal{T}(\Lambda_1)^{\otimes N}$, where $N$ is the number of boxes in $T$. Then apply the tensor product rule to obtain $f_a T$ and $e_a T$, $a \in I$. With these crystal operations, $\mathcal{T}(\infty) \cong B(\infty)$ as $U_q(\mathfrak{g})$-crystals in types $A_n$, $B_n$, $C_n$, $D_{n+1}$, and $G_2$, as shown in [9].

Now, we will construct a bijection $\Psi\colon \mathrm{RC}(\infty) \longrightarrow \mathcal{T}(\infty)$. Define

$$\varpi_i = \begin{cases} 2\Lambda_n & \text{if } \mathfrak{g} = B_n \text{ and } i = n, \\ \Lambda_n + \Lambda_{n+1} & \text{if } \mathfrak{g} = D_{n+1} \text{ and } i = n, \\ \Lambda_i & \text{otherwise,} \end{cases}$$

and note that $\varpi_i$ will directly correspond to a column of height $i$. (We do not consider $\varpi_{n+1}$ in type $D_{n+1}$.) For a sequence of partitions $\nu = (\nu^{(a)})_{a \in I}$, define $\lambda_\nu \in P^+$ by $\lambda_\nu := \sum_{a=1}^{n-1} \bigl(|\nu^{(a)}| + 1\bigr)\varpi_a + \lambda_\nu^{(n)}$, where

$$\lambda_\nu^{(n)} := \begin{cases} \bigl(\max\bigl(|\nu^{(n)}|, |\nu^{(n+1)}|\bigr) + 1\bigr)\varpi_n & \mathfrak{g} = D_{n+1}, \\ \bigl(|\nu^{(n)}| + 1\bigr)\varpi_n & \text{otherwise.} \end{cases}$$

We define a map $\delta_k\colon \mathrm{RC}(\lambda) \longrightarrow \mathrm{RC}(\lambda - \varpi_k)$ as follows. Take the longest path

$$\boxed{k} = \boxed{\pi_0} \xrightarrow{a_0} \boxed{\pi_1} \xrightarrow{a_1} \cdots \xrightarrow{a_m} \boxed{\pi_{m+1}}$$

in $\mathcal{T}(\Lambda_1)$ such that there exists $0 \le \ell_{\pi_0} \le \cdots \le \ell_{\pi_m}$ and singular, or quasisingular if $\pi_{i+1} = 0$, rows of length $\ell_{\pi_i}$ in $\nu^{(a_i)}$ for all $0 \le i \le m$.[3] For such a path, take the

[3]For types $C_n$ and $D_n$, the rows must be distinct, and for type $G_2$, the rows must be distinct if $\pi_0 = 1$ and $\pi_2 = 3$.

| $\mathfrak{g}$ | $A_n$ | $B_n$ | $C_n$ | $D_{n+1}$ | $G_2$ |
|---|---|---|---|---|---|
| $\widetilde{\mathfrak{g}}$ | $A_n^{(1)}$ | $D_{n+1}^{(2)}$ | $A_{2n-1}^{(2)}$ | $D_{n+1}^{(1)}$ | $D_4^{(3)}$ |

**Table 6.1:** The association of affine type $\widetilde{\mathfrak{g}}$ with a classical type $\mathfrak{g}$ used here.

smallest possible $0 \leq \ell_{\pi_0} \leq \cdots \leq \ell_{\pi_m}$ and remove a box from each (quasi)singular row of length $\ell_{\pi_i}$. Then make the resulting rows singular, also make the row of length $\ell_{\pi_i}$ quasisingular if $\pi_i = 0$ and the row of length $\ell_{\pi_{i-1}}$ is quasisingular. The result is $\delta_k(\nu, J)$, and we say $\delta_k$ returns $\pi_{m+1}$.

Now for $\lambda_\nu = \sum_{a=1}^n c_a \varpi_a$, we define $\Psi(\nu, J)$ first by considering the return values of $\delta_1^{c_1} \circ \delta_2^{c_2} \circ \cdots \circ \delta_n^{c_n}(\nu, J)$. We construct a (large) tableaux $T$ column-by-column from left-to-right as inserting the column whose reading word is $12 \cdots (k-1)r$, where $r$ is the return value of $\delta_k$. We then define $\Psi(\nu, J)$ as the reduction of $T$ to its marginally large representative.

**Theorem 6.2** ([26])**.** *The map* $\Psi$*, defined above, is a crystal isomorphism.*

We will now sketch a proof of Theorem 6.2, we need to consider an affine type $\widetilde{\mathfrak{g}}$ whose classical subalgebra is $\mathfrak{g}$. However we do not do so in the usual fashion by taking the untwisted affine algebra, but instead consider those given by Table 6.1. We also require certain $U_q'(\widetilde{\mathfrak{g}}) := U_q([\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}])$ crystals called Kirillov–Reshetikhin (KR) crystals $B^{r,1}$ [6], where $r$ is a node in the Dynkin diagram of $\mathfrak{g}$.

In [20], a bijection $\Phi$ from classically highest weight elements in a tensor product of KR crystals of the form $(B^{1,1})^{\otimes N}$ for all non-exceptional affine types was described. A similar bijection in type $D_4^{(3)}$ was given in [33]. This was also extended to tensor products of the form $\bigotimes_{i=1}^N B^{r_i,1}$ [22, 29, 31, 33].

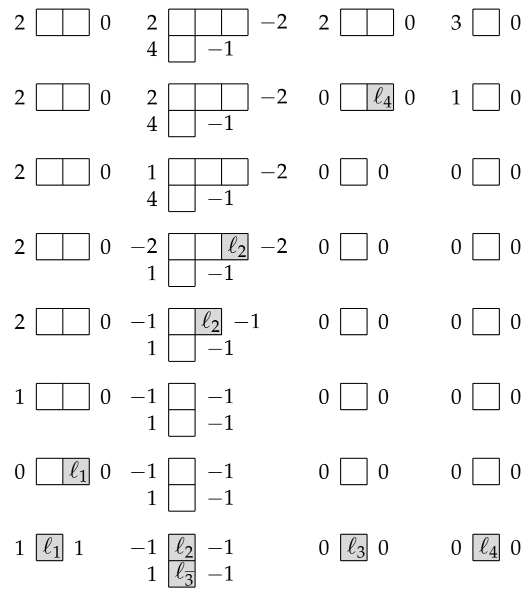
Let $B^{\otimes \lambda} = \bigotimes_{a=1}^n \left(B^{a,1}\right)^{\otimes \langle h_a, \lambda \rangle}$, and denote the corresponding $U_q'(\widetilde{\mathfrak{g}})$-rigged configurations by $\mathrm{RC}(B^{\otimes\lambda})$ [21, 22, 31]. We note that there is a unique classical component isomorphic to $B(\lambda) \subseteq B^{\otimes\lambda}$. Hence, there is a natural injection of $\mathrm{RC}(\lambda_\nu)$ into $\mathrm{RC}(B^{\otimes\lambda_\nu})$. Let $\mathcal{T}(\lambda_\nu)$ denote the subcrystal of $\mathcal{T}(\infty)$ whose shape fits inside of $\lambda_\nu$. Therefore, we can define $\Psi \colon \mathrm{RC}(\infty) \longrightarrow \mathcal{T}(\infty)$ using the composition

$$
\begin{array}{ccccccccc}
\mathrm{RC}(\infty) & \twoheadrightarrow & \mathrm{RC}(\lambda_\nu) & \hookrightarrow & \mathrm{RC}(B^{\otimes\lambda_\nu}) & \twoheadrightarrow & \mathcal{T}(\lambda_\nu) & \hookrightarrow & \mathcal{T}(\infty), \\
(\nu, J) & \longmapsto & (\nu, J) & \longmapsto & (\nu, J) & \longmapsto & \Phi(\nu, J) & \longmapsto & \Phi(\nu, J).
\end{array}
$$

**Example 6.3.** *Consider the rigged configuration* $(\nu, J)$ *from Example 4.2. We have* $\lambda_\nu = 3\Lambda_1 + 5\Lambda_2 + 3(\Lambda_3 + \Lambda_4)$*. So projecting* $(\nu, J)$ *to* $\mathrm{RC}(\lambda_\nu)$ *and then to* $\mathrm{RC}(B^{\otimes\lambda_\nu})$*, we have*

$$
2\ \square\square\ 0 \qquad \begin{array}{l} 2\ \square\square\square\ -2 \\ 4\ \square\ -1 \end{array} \qquad 2\ \square\square\ 0 \qquad 3\ \square\ 0\ .
$$

*Next, we perform $\Phi$ (performing all trivial steps simultaneously):*



$$3,3 \rightsquigarrow \Lambda_3 + \Lambda_4,$$
$$4 \rightsquigarrow \Lambda_3 + \Lambda_4,$$
$$2,2,2 \rightsquigarrow \Lambda_2,$$
$$3 \rightsquigarrow \Lambda_2,$$
$$3 \rightsquigarrow \Lambda_2,$$
$$1 \rightsquigarrow \Lambda_1,$$
$$2 \rightsquigarrow \Lambda_1,$$
$$\overline{2} \rightsquigarrow \Lambda_1,$$

*Thus, we have*

$$\Psi(\nu, J) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 1&1&1&1&1&1&1&1&1&2&\overline{2} \\ \hline 2&2&2&2&2&2&2&3&3 \\ \cline{1-9} 3&3&4 \\ \cline{1-3} \end{array} \equiv \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1&1&1&1&1&1&2&\overline{2} \\ \hline 2&2&2&3&3 \\ \cline{1-5} 3&4 \\ \cline{1-2} \end{array} .$$

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